Model for Relating Coupled Power Equations to Coupled Amplitude Equations

By D. T. YOUNG

(Manuscript received July 24, 1963)

Following a suggestion of H. E. Rowe, the first-order system of coupled differential equations with the independent variables representing amplitudes of coupled modes (the so-called coupled mode equations) has been rigorously transformed into similar equations with the independent variables representing the power in the coupled modes for the case where the amplitude coupling coefficient is a Gaussian random process with a white noise spectrum and zero mean value. Coupled mode equations in power have proven useful, and this work provides formal justification and identification of the variables and parameters.

The equations may allow one to approach coupled-mode problems directly for statistical results.

I. INTRODUCTION

Consider the coupled line equations:

$$I_0'(z) = -\Gamma_0 I_0(z) + jc(z) I_1(z) \tag{1}$$

$$I_1'(z) = jc(z) I_0(z) - \Gamma_1 I_1(z). \tag{2}$$

These equations are useful in describing effects of coupling between a signal mode, represented by a complex wave amplitude $I_0(z)$, and a single spurious mode, represented by $I_1(z)$, caused by geometric imperfections in a multimode transmission line. These equations may be derived in two ways from basic principles:[1,2,3,4] direct conversion of Maxwell's equations to generalized telegraphist's equations, or allowing discrete converters to become continuous. Exact solutions are known in only a few special cases, one of which is the case of constant $c(z)$. Making use of this solution and assuming $c(z)$ is a stationary Gaussian random function with a white noise spectrum S_0 and zero mean value, it is

possible to derive coupled differential equations with the expected values of the power in $I_0(z)$ and $I_1(z)$ as independent parameters. That is

$$P_0'(z) = (-2\alpha_0 - S_0)P_0(z) + S_0P_1(z) \tag{3}$$

$$P_1'(z) = S_0P_0(z) + (-2\alpha_1 - S_0)P_1(z) \tag{4}$$

where

$$P_{\substack{0\\1}}(z) = \langle I_{\substack{0\\1}}(z) I_{\substack{0\\1}}^*(z)\rangle.$$

These equations are similar to equations used by S. E. Miller.[5]

II. DESCRIPTION OF MATHEMATICAL MODEL AND COMPUTATION

Consider (1) and (2) and assume the $I_{\substack{0\\1}}(z)$ are known and $c(z)$ has the constant value c in $(z, z + \Delta z)$; then $I_{\substack{0\\1}}(z + \Delta z)$ are given by[4]

$$\begin{bmatrix} I_0(z + \Delta z) \\ I_1(z + \Delta z) \end{bmatrix} = (e^{-\Gamma_0 \Delta z})\mathbf{G} \begin{bmatrix} I_0(z) \\ I_1(z) \end{bmatrix}$$

where $\mathbf{G}$ is a two-by-two matrix with elements

$$g_{11} = \left(\frac{a + j1}{2a}\right) \exp\left[\frac{\Delta\Gamma}{2}(1 + ja)\Delta z\right] + \left(\frac{a - j1}{2a}\right) \exp\left[\frac{\Delta\Gamma}{2}(1 - ja)\Delta z\right]$$

$$g_{22} = \left(\frac{a - j1}{2a}\right) \exp\left[\frac{\Delta\Gamma}{2}(1 + ja)\Delta z\right] + \left(\frac{a + j1}{2a}\right) \exp\left[\frac{\Delta\Gamma}{2}(1 - ja)\Delta z\right]$$

$$g_{12} = g_{21} = \left(\frac{c}{a\Delta\Gamma}\right) \exp\left[\frac{\Delta\Gamma}{2}(1 + ja)\Delta z\right] - \left(\frac{c}{a\Delta\Gamma}\right) \exp\left[\frac{\Delta\Gamma}{2}(1 - ja)\Delta z\right]$$

where

$$a = \sqrt{\left(\frac{2c}{\Delta\Gamma}\right)^2 - 1}$$

$$\Delta\Gamma = \Gamma_0 - \Gamma_1 = \Delta\alpha + j\Delta\beta.$$

Now compute $\langle I_0 I_0^* \rangle$ and $\langle I_1 I_1^* \rangle$ from

$$\begin{bmatrix} I_0 I_0^* \\ I_0 I_1^* \\ I_1 I_0^* \\ I_1 I_1^* \end{bmatrix}_{z+\Delta z} = (e^{-2\alpha_0 \Delta z})(\mathbf{G} \times \mathbf{G}^*) \begin{bmatrix} I_0 I_0^* \\ I_0 I_1^* \\ I_1 I_0^* \\ I_1 I_1^* \end{bmatrix}_z$$

where $\times$ denotes the Kronecker product (direct product) of the matrices $\mathbf{G}$ and $\mathbf{G}^*$.[6] If we compute ensemble averages, assuming the random function $c(z)$ is independent of previous values and $\langle c(z) \rangle = 0$, we have

$$P_0(z + \Delta z) = e^{-2\alpha_0 \Delta z}[AP_0(z) + BP_1(z)] \tag{5}$$

$$P_1(z + \Delta z) = e^{-2\alpha_0 \Delta z}[BP_0(z) + DP_1(z)] \tag{6}$$

where

$$P_{\substack{0\\1}}(z) = \langle I_{\substack{0\\1}}(z) I_{\substack{0\\1}}^*(z) \rangle$$

$$A = \langle g_{11} g_{11}^* \rangle$$

$$B = \langle g_{12} g_{12}^* \rangle$$

$$D = \langle g_{22} g_{22}^* \rangle$$

since $\langle g_{11} g_{12}^* \rangle = \langle g_{12} g_{11}^* \rangle = \langle g_{12} g_{22}^* \rangle = \langle g_{22} g_{12}^* \rangle = 0$, being odd functions of c. Expanding in powers $\langle c^2 \rangle$ and Δz and requiring

$$\lim_{\substack{\langle c^2 \rangle \to \infty \\ \Delta z \to 0}} \langle c^2 \rangle \Delta z = S_0$$

and after considerable algebraic manipulation, which we omit, we obtain (3) and (4)

$$P_{\substack{0\\1}}(z) = (-2\alpha_{\substack{0\\1}} - S_0) P_{\substack{0\\1}}(z) + S_0 P_{\substack{1\\0}}(z).$$

These equations may be related to Miller's equations by replacing our P_0, P_1, $2\alpha_0$, $2\alpha_1$, and S_0 by his $P_1 + P_n$, P_x, $a_{1h} + a_{nh}$, a_{xh}, and a_{1x}.

III. ACKNOWLEDGMENT

The author would like to thank H. E. Rowe, who suggested this model and approach.

REFERENCES

1. Miller, S. E., Coupled Wave Theory and Waveguide Applications, BSTJ, **33**, May, 1954, pp. 661–720.
2. Schelkunoff, S. A., Conversion of Maxwell's Equations into Generalized Telegraphist's Equations, BSTJ, **34**, September, 1955, pp. 995–1043.
3. Morgan, S. P., Theory of Curved Circular Waveguide Containing an Inhomogenous Dielectric, BSTJ, **36**, September, 1957, pp. 1209–1251.
4. Rowe, H. E., and Warters, W. D., Transmission in Multimode Waveguides with Random Imperfections, BSTJ, **41**, May, 1962, pp. 1062–1068.
5. Miller, S. E., Waveguide as a Communication Medium, BSTJ, **33**, November, 1954, pp. 1209–1265.
6. Bellman, Richard, *Introduction to Matrix Analysis*, McGraw-Hill Book Co., Inc., New York, 1960.